Singular monotonic functions defined by a convergent positive series and a double stochastic matrix

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Let

1) $1 = \frac{2}{3} + \sum\limits_{n=1}^{\infty} \frac{1}{(-2)^n} = \frac{2}{3} - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$ be normalized alternating binary series that defines a binary negapositional image of the number of the segment $[0;1]$:

$$x = \frac{2}{3} + \frac{\alpha_1(x)}{(-2)^1} + \frac{\alpha_2(x)}{(-2)^2} + \frac{\alpha_3(x)}{(-2)^3} + \ldots + \equiv \overline{\Delta}^2_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots};$$

2) $\|p_{ik}\| = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ be a positive double stochastic matrix i.e. $p_{ij} > 0,\ p_{i0} + p_{i1} = 1,\ p_{0j} + p_{1j} = 1,\ i = 0, 1,\ j = 0, 1;$

3) $\overline{p} = (p_0; p_1)$ be a vector $p_0 = \frac{p_{10}}{p_{01}+p_{10}} = \frac{1}{2}$ and $p_1 = \frac{p_{01}}{p_{01}+p_{10}} = \frac{1}{2}$;

It is known that a binary negapositional number representation is a recoding of a classical binary representation:

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_n}{2^n} + \ldots \equiv \Delta^2_{a_1(x)a_2(x)\ldots a_n(x)\ldots}, \quad a_n \in \{0;1\}$$

Considered in the talk is function F, defined by equality

$$F(x) = F(\overline{\Delta}^2_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}) = \beta_{\alpha_1(x)} + \frac{1}{2}\sum_{k=1}^{\infty} (\beta^{(k)}_{\alpha_k(x)\alpha_{k+1}(x)} \prod_{i=1}^{k-1} p_{\alpha_i(x)\alpha_{i+1}(x)}), \quad \textit{where} \tag{1}$$

$$\beta_{\alpha_1(x)} = \begin{cases} 0, & \text{if} \quad \alpha_1(x) = 1, \\ \frac{1}{2}, & \text{if} \quad \alpha_1(x) = 0, \end{cases}$$

$$\beta^{(2n-1)}_{\alpha_{2n-1}(x)\alpha_{2n}(x)} = \beta^{(1)}_{\alpha_{2n-1}(x)\alpha_{2n}(x)} = \begin{cases} 0, & \text{if} \quad \alpha_{2n}(x) = 0, \\ p_{00}, & \text{if} \quad \alpha_{2n-1}(x) \neq \alpha_{2n}(x) = 1, \\ p_{10}, & \text{if} \quad \alpha_{2n-1}(x) = \alpha_{2n}(x) = 1, \end{cases}$$

$$\beta^{(2n)}_{\alpha_{2n}(x)\alpha_{2n+1}(x)} = \beta^{(0)}_{\alpha_{2n}(x)\alpha_{2n+1}(x)} = \begin{cases} 0, & \text{if} \quad \alpha_{2n+1}(x) = 1, \\ p_{01}, & \text{if} \quad \alpha_{2n}(x) = \alpha_{2n+1}(x) = 0, \\ p_{00}, & \text{if} \quad \alpha_{2n}(x) \neq \alpha_{2n+1}(x) = 0, \end{cases}$$

and $\alpha_k(x)$ is k negapositional digit of representation of the number x.

Definition 1. Let $(c_1, c_2, \ldots, c_m)$ be a orderly set of positive integers. *The Cylinder of m rank with basis $c_1c_2\ldots c_m$* is called a set $\overline{\Delta}^2_{c_1c_2\ldots c_m}$ of numbers of $x \in (0;1]$ that is first m negapositional digits of which are $c_1, c_2, \ldots, c_m$ respectively, i.e.

$$\overline{\Delta}^2_{c_1c_2\ldots c_m\ldots} = \left\{x:\ x = \overline{\Delta}^2_{c_1c_2\ldots c_m a_{m+1}a_{m+2}\ldots}, \quad a_{m+i} \in \mathbb{N},\ i = 1, 2, 3, \ldots\right\}.$$

Lemma 2. *For a function F defined by the equality (1) the mapping of the cylinder $\overline{\Delta}^2_{c_1c_2\ldots c_m}$ is a segment $[a;b]$, where*

$$a = \beta_{c_1} + \frac{1}{2}\sum_{k=1}^{m-1} \left(\beta^{(k)}_{c_kc_{k+1}} \prod_{j=1}^{k-1} q_{c_jc_{j+1}}\right), \qquad b = a + \frac{1}{2}\prod_{j=1}^{m-1} q_{c_jc_{j+1}},$$

Theorem 3. *Images of different cylinders of the same rank with the mapping F do not overlap and in the union give the whole segment* $[0, 1]$.

Theorem 4. *The function $F(x)$ denoted by the equality (1) is:*

1) correctly identified,

2) continuous,

3) strictly increasing,

4) linear for $p_{00} = 0.5$ and singular for $p_{00} \neq 0.5$ (has a derivative equal to zero almost everywhere in the sense of the Lebesgue measure).

The report proposes the results of studies of the above-mentioned functions.

References

[1] *Працьовитий М. В.* Фрактальний підхід у дослідженнях сингулярних розподілів. — Київ: Вид-во НПУ імені М. П. Драгоманова, 1998. — 296 с.